Caratheodory convergence

Geometric characterization of convergence in (S).

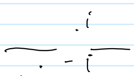
Def. Let Ω_n be a sequence of S.C. domains. $w_0 \in \Omega_n \, \forall n$.
The kernel of Ω_n with respect to w_0 is:
$\{w_0\} \cup \{w \in \mathbb{C} : \exists \text{ domain } \mathcal{H} : w_0 \in \mathcal{H}, w \in \mathcal{H}, \exists N : \mathcal{H} \subset \Omega_n, n > N\}$.
No such w - kernel is $\{w_0\}$.

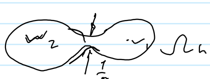
Otherwise: $\ker(\{\Omega_n\}, w_0)$ - domain $=: \Omega$. $w_0 \in \Omega$.

Def. $(\Omega_n, w_0) \xrightarrow{\ker} (\Omega, w_0)$ if $\forall n_k$ - subsequence, $\ker((\Omega_{n_k}), w_0) = \Omega$.

Example. Let $\Omega_{n+1} \subset \Omega_n$. $\Omega := \operatorname{int}(\bigcap \Omega_n)$ - component containing w_0.
Then $\Omega_n \xrightarrow{\ker} \Omega$ if $\Omega \neq \emptyset$, and $\Omega_n \xrightarrow{\ker} \{w_0\}$ otherwise!

Pf By monotonicity, $\ker((\Omega_{n_k}), w_0) = \ker(\Omega_n, w_0)$. So need: $\Omega = \ker$.
$\Omega \subset \ker$ (since can take $\mathcal{H} = \Omega$).
On the other hand: $w \in \ker \Rightarrow \exists \mathcal{H} \subset \Omega_n \, \forall n$ (monotonicity!) $\Rightarrow \mathcal{H} \subset \bigcap \Omega_n$.
so $\mathcal{H}$ - open, $w_0 \in \mathcal{H}$ $\Rightarrow \mathcal{H} \subset \Omega. \Rightarrow w \in \Omega$ #

Examples (to example). 1)  $\Omega_n = \mathbb{C} \setminus \{(-\infty, -\frac{1}{n}] \cup [\frac{1}{n}, \infty)\}$.
$(\Omega_n, i) \to (\mathbb{H}, i)$ $(\Omega_n, -i) \to (\overline{\mathbb{H}}, -i)$

2)  Ω_n $(\Omega_n, w_1) \to$  $(\Omega_n, w_2) \to$ 

Equivalent def $\Omega_n \xrightarrow{\ker} \Omega$; $w_0 \in \Omega_n$, $w_0 \in \Omega$ iff
1) $\forall K$ - compact $K \subset \Omega \Rightarrow \exists N : K \subset \Omega_n \; \forall n > N$.
2) $\forall c \in \partial\Omega \;\; \exists c_n \in \partial\Omega_n : c_n \to c$.

(2) can be restated $\operatorname{dist}(c, \partial\Omega_n) \to 0$, $\forall c \in \partial\Omega$)

Pf (of equivalency).

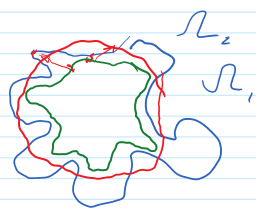
(I). Let $\Omega_n \xrightarrow{ker} \Omega$. $K \subset \Omega$ - compact $\forall x \in K\ \exists\, \mathcal{U}_x$ - domain, $x \in \mathcal{U}_x$,
ker $\Rightarrow$ $w_0 \in \mathcal{U}_x$. $\bigcup \mathcal{U}_x \supset K \underset{compact}{\Rightarrow} K \subset \bigcup_{l=1} \mathcal{U}_l \subset \Omega_n$ for large enough n. $\Rightarrow$ 1).

Now take $c \in \partial\Omega$. If $\not\exists\, c_n$, then $\exists\, \varepsilon>0,\ n_k \uparrow \infty$: $B(c,\varepsilon)\cap\partial\Omega_{n_k}=\emptyset$.
Take $w\in B(c,\varepsilon)\cap\Omega$. $\cancel{\Omega}$ = ker $\Rightarrow$ $w\in \mathcal{U}\subset\Omega_n$ for large n,
so $\mathcal{U}\cup B(c,\varepsilon)\subset\Omega_{n_k}$ for large k $\Rightarrow$ $c \in ker(\Omega_{n_k})$ - contradiction!

(II) $\Rightarrow$ Ker. $w\in\Omega \Rightarrow \exists\, K\subset\Omega: \{w,w_0\}\subset Int\, K$, Int K - connected
Then $K\subset\Omega_n$ for large n. So Int $K\subset\Omega_n$, take $\mathcal{U}$ = Int K $\Rightarrow$ $\Omega\subset ker(\Omega_n,w_0)$.
On the other hand, $c\in\partial\Omega \Rightarrow c\notin ker(\Omega_{n_k})$ ($c_{n_k}\to c$,
$\mathcal{U}\ni c \Rightarrow c_{n_k}\in\mathcal{U}$ for large k - contradiction with $\mathcal{U}\subset\Omega_{n_k}$).
So $\partial\Omega\subset\mathbb{C}\setminus Ker(\Omega_{n_k})\ \forall(n_k) \Rightarrow \Omega = ker(\Omega_{n_k})$ ∎

Yet another equivalent definition (common interior approximation).

$K\subset\Omega_1\cap\Omega_2$ - connected compact is called common ε-interior approximation if
1) $w_0\in K$
2) $\forall w\in\partial K\ \exists\, w_1\in\partial\Omega_1,\ w_2\in\partial\Omega_2$: $|w-w_1|<\varepsilon$, $|w-w_2|<\varepsilon$.



$\Omega_n \xrightarrow{int.\ approx} \Omega$ if $\forall\varepsilon>0\ \exists\, K\subset\Omega$ and N: K is a common ε-int approximation for Ω_n and Ω, $\forall n\ge N$.

Pf. (ker $\Rightarrow$ Int). Take K_ε - an $\varepsilon/2$-int. approx. to Ω. K_ε - compact $\Rightarrow$ $K_\varepsilon\subset\Omega_n\ \forall n\ge N$. Take $z\in\partial K_\varepsilon$ and $c\in\partial\Omega: |c-z|<\frac{\varepsilon}{2}$. If $\exists\, n_k\uparrow\infty$: dist$(z,\partial\Omega_{n_k})>\varepsilon$, then dist$(c,\partial\Omega_n)>\frac{\varepsilon}{2}$ + contradiction.

(Int $\Rightarrow$ ker) Let $F\subset\Omega$ - compact,
Take $\tilde F\supset F\cup\{w_0\}$ - connected compact. Take $\varepsilon <$ dist$(\tilde F,\partial\Omega)$.
Then $\tilde F\subset K_\varepsilon$, so $K\subset\Omega_n$ $n\ge N$. Thus $\Omega\subseteq ker(\Omega_n,w_0)$.

If $w\in ker(\Omega_{n_k}))\setminus\Omega$, then $\exists\, \mathcal{U}\subset\bigcap\Omega_{n_k}$, $\mathcal{U}\ni w,w_0$ - open.
Observe: $\Omega=\bigcup K_\varepsilon$. Then $\mathcal{U}\cup\bigcup K_\varepsilon\supset\Omega$, $\Omega_{n_k}\supset\mathcal{U}$ $\Rightarrow$
$\forall z\in\partial K_\varepsilon$: dist$(z,\partial\mathcal{U})<\varepsilon$. $w\in\mathcal{U}\setminus\Omega$. $\exists\, F\subset\mathcal{U}$ - connected compact
$w,w_0\in F$. Let 2ε = dist$(F,\partial\mathcal{U})$. Then any ε-int. approx. of $\mathcal{U}\subset\Omega_n$ contains F.
Thus $K_\varepsilon\ni w$ - contradiction ∎

Constantin Carathéodory
(1873-1950)

Theorem (Caratheodory). Let $f_n : \mathbb{D} \to \Omega_n$ - univalent, $f_n(0) = w_0$, $f_n'(0) > 0$.
$f : \mathbb{D} \to \Omega$ - $f(0) = w_0$, $f'(0) > 0$

Then $f_n(z)$ converges locally uniformly to $f \iff \Omega_n \xrightarrow{ker} \Omega$, $\Omega \neq \mathbb{C}$.

Pf (I) $f_n \to f \implies \Omega = ker(\Omega_n)$.

(Since also $f_{n_k} \to f$, we have $\Omega = ker(\Omega_{n_k})$. So it is enough for this direction. Also, since $f \not\equiv$ const, f is univalent and $\Omega := f(\mathbb{D}) \neq \mathbb{C}$)

Let $w \in \Omega$. Need: $w \in ker(\Omega_n)$. If $f \equiv w_0$, nothing to prove. Let $w = f(z)$. Take r: $|z| < r < 1$. $\mathcal{H} := \{f(z) : |z| < r\}$.
Need: $\mathcal{H} \subset \Omega_n$ for large n. Assume not. $\exists n_k \to \infty$, $w_k \in \mathcal{H}$: $w_k \notin \Omega_{n_k}$.

Pass to a subsequence to assume $w_{n_k} \to w^*$

Use Rouche: $f_{n_k}(z) - w_{n_k} \to f(z) - w^*$ and $f_{n_k}(z) \neq w_{n_k} \implies$
$f(z) \neq w^*$ $\forall z \in \mathbb{D}$.

But $w^* \in \overline{\mathcal{H}} \subset \Omega$ - contradiction.

Let $w \in ker(\Omega_n)$. Take $\mathcal{H} \ni (w, w_0)$ connected. For large n, $\mathcal{H} \subset \Omega_n$, so
$g_n (= f_n^{-1})$ - univalent on $\mathcal{H}$, $g_n(w_0) = 0$, $g_n'(0) > 0$. $g_n(\mathcal{H}) \subset \mathbb{D}$. By Montel, $\exists n_k$.
$g_{n_k} \to g$. $g(0) = 0$, $|g(w)| \leq 1$ in $\mathcal{H}$. By Maximum Principle, $g(\mathcal{H}) \subset \mathbb{D}$.
f_{n_k} converges locally uniformly near $z := g(w)$. Then $g_{n_k}(w) \to g(w)$ and
$f_{n_k}(g_{n_k}(w)) = w$, we get $w = f(z)$ so $w \in \Omega$ ∎

(II) Let $\Omega_n \xrightarrow{\text{ker}} \Omega$, $\Omega \neq \mathbb{C}$.

First: (f_n) is normal.

Know: $\{w : |w - w_0| < \frac{1}{4} f_n'(0)\} \subset \Omega_n$. If $f_n'(0)$ - unbounded $\Rightarrow$ $\exists (\Omega_{n_k})$: $\mathbb{C} = \ker(\Omega_{n_k})$.
(by Koebe)

So $|f_n'(0)|$ is bounded.

Also $|f_n(z)| \le |f_n'(0)| \frac{|z|}{(1-|z|)^2}$, so $|f_n(z)|$ is locally bounded $\Rightarrow$ normal.
(Since $\frac{f_n(z)}{f_n'(0)} \in (S)$)

Assume $f_n \not\to f$. Then, by normality, $\exists f_{n_k} \to f^* \neq f$.

By (I), $\Omega_{n_k} \to \Omega^* = f^*(\mathbb{D})$. But $\Omega_{n_k} \to \underset{f(\mathbb{D})}{\underset{\shortparallel}{\Omega}}$ - contradiction!